Microwave Heating of a Luneberg Lens

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A calculation is made of the maximum steady-state temperature rise due to a small amount of dielectric dissipation in a Luneberg lens which is continuously illuminated by a powerful microwave transmitter, when the surface of the lens is held at a constant temperature. The temperature distribution along the axis of the lens is computed both when the focus is at the surface, and also when it is outside the surface at a distance equal to one-tenth of the lens radius. The numerical results are given in such a form that the maximum temperature rise is easily deduced when the loss tangent is any linear function of the refractive index of the lens material. In general the maximum steady-state temperature occurs on the axis at some interior point between the center of the lens and the focus. The total power dissipated in the lens is also computed. Finally, a brief discussion is given of the time scale associated with transient heating of the lens.

I. INTRODUCTION AND SUMMARY

When a Luneberg lens is to be used as an antenna for a long-range radar,[1] it is important to know how much the lens will be heated by dielectric dissipation when it is illuminated by a powerful transmitter. This paper presents a calculation of the maximum steady-state temperature rise in the interior of the lens, when the surface is held at a constant temperature and only a small fraction of the incident power is dissipated. Since the maximum temperature rise depends critically on the loss tangent of the lens material, and since the loss tangent may vary with index of refraction, the results are given in such a form that the maximum temperature rise is easily deduced when the loss tangent is any linear function of the index of refraction. The index of refraction is assumed to vary with radius in a manner appropriate for a Luneberg lens of the desired focal length. Numerical computations have been made for the case in which the focus is at the surface of the lens, and also when it is outside the surface at a distance equal to one-tenth of the lens radius.

In the idealized problem to be studied, the temperature rise per watt of incident power will be the same for a distant transmitter, which illuminates the lens with an essentially uniform plane wave, as for a transmitter at the focus, with a feed pattern corresponding to a uniform plane emergent wave. In either case, if the lens is illuminated from a particular direction, and if the index of refraction and the loss tangent are functions of the radial coordinate only, the heat source distribution and the temperature distribution will be axially symmetric, and in all practical cases the maximum temperature will occur somewhere on the axis of the lens. Hence to find the maximum temperature we need only calculate the temperature distribution along the axis.

Suppose that the loss tangent of the lens material can be adequately represented by a linear function of the index of refraction; thus

$$\tan \delta = An + B, \tag{1}$$

where n is the refractive index and A and B are constants. Then we shall show that the axial temperature distribution can be written in the form

$$T(\xi) = (P_0/k\lambda)[AT_A(\xi) + BT_B(\xi)], \tag{2}$$

where ξ is axial distance in units of the lens radius, with $\xi = -1$ corresponding to the plane wave side and $\xi = +1$ to the side nearest the focus. P_0 is the total power incident on the lens, k is the thermal conductivity, and λ is the free-space wavelength. The dimensionless functions $T_A(\xi)$ and $T_B(\xi)$ are given in Table I of Section V and are plotted in Figs. 2 and 3 for lenses with normalized focal distances of 1.0 and 1.1, measured from the center. Note that the maximum temperature given by (2) is independent of the lens radius.

The foregoing remarks apply to the case in which the lens is illuminated from a single direction, so that the maximum temperature rise occurs on the axis. If the total power P_0 striking the lens comes from several different directions, we can deduce upper and lower bounds on the maximum temperature rise in the "multiaxial" case from a knowledge of the temperature distribution along the axis in the "uniaxial" case. Since the heat conduction equation is linear, the principle of superposition guarantees that the temperature at the center of the lens is the same in both cases. Also, the maximum temperature in the multiaxial case is less than the maximum temperature in the uniaxial case, since the maximum temperature point in the uniaxial case is on the axis defined by the incident beam, and this point is not on the other

axes in the multiaxial case. It follows that the maximum temperature rise in the multiaxial case is at least as great as the rise at the center of the lens in the uniaxial case, but not as great as the maximum rise in the uniaxial case. The numerical example in Section V indicates that in practical situations these two bounds may be so close together that a more detailed treatment of the multiaxial case would be superfluous.

The body of the paper is concerned with the determination of the functions $T_A(\xi)$ and $T_B(\xi)$ which occur in (2). In Section II we introduce a convenient approximation to the index of refraction of the Luneberg lens, which is exact if the focus is at the surface, and show that under this approximation the ray paths are ellipses. In Section III we compute the power flow through every element of a lossless lens, and the approximate rate of dissipation of heat, assuming small dissipation and a loss tangent of the form (1). An integral representation of the temperature along the axis is obtained in Section IV, as well as an expression for the total dissipated power. Results of numerical integrations carried out on an IBM 7090 are given in Section V. Appendix A contains a proof that in all practical cases the maximum "uniaxial" temperature occurs on the axis, while Appendix B is concerned with the nature of the mathematical singularity which occurs in the idealized model when the focal point is at the surface of the lens. The time scale for transient thermal effects is briefly discussed in Appendix C.

II. RAY PATHS IN A LUNEBERG LENS

The path of a typical ray in a Luneberg lens of normalized radius unity is shown schematically in Fig. 1. In general the path of a light

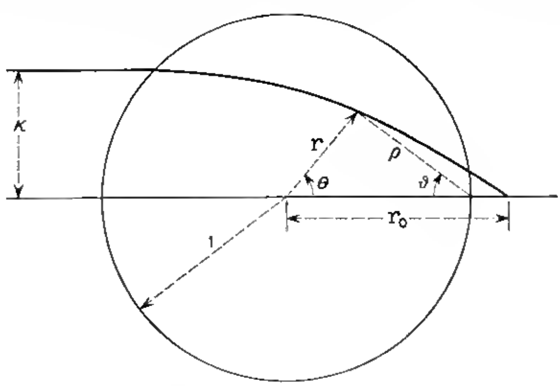

Fig. 1 — Typical ray path in a Luneberg lens.

ray in a radially symmetric medium with refractive index $n(r)$ is given by[2]

$$\theta - \theta_0 = \pm \int_{r_0}^{r} \frac{\kappa\, dr}{r[n^2r^2 - \kappa^2]^{\frac{1}{2}}}, \tag{3}$$

where (r,θ) are polar coordinates in the plane of the ray, and κ is constant for a particular ray, being determined, together with the ambiguous sign, by the direction of the ray at the initial point (r_0, θ_0). In the special case where the rays are all initially parallel, κ is equal to the initial distance of the given ray from the axis of the system.

In principle, (3) may be integrated to find the ray path whenever n is a known function of r. For a Luneberg lens with focal point offset from the surface, however, the relationship between n and r is given by two parametric equations[2] involving a function defined by a definite integral, and it does not seem possible to obtain the equation of the ray path explicitly in terms of known functions. An approximation to the refractive index which does permit analytic integration of (3) is

$$n = [n_0^2 - (n_0^2 - 1)r^2]^{\frac{1}{2}}, \tag{4}$$

where n_0 is the index at the center of the lens according to the accurate theory; n_0 is a decreasing function of the focal length r_0. The relationship (4) is exact if the focus is at the surface ($n_0 = \sqrt{2}$), and is a good approximation if the distance from the focus to the surface is small.

To find the equation of a typical ray in a lens whose refractive index is given by (4), it is convenient first to locate the "turning point" (r^*, θ^*) at which the distance of the ray from the center of the lens is a minimum. The turning point is defined by[2]

$$r^* n(r^*) = \kappa, \tag{5}$$

which yields, using (4),

$$r^*(\kappa) = \left[\frac{n_0^2 - [n_0^4 - 4\kappa^2(n_0^2 - 1)]^{\frac{1}{2}}}{2(n_0^2 - 1)}\right]^{\frac{1}{2}}. \tag{6}$$

The corresponding angle is derived from (3), setting $\theta = \pi$ when $r = \infty$ and noting that θ and r decrease together. We obtain

$$\begin{aligned} \theta^*(\kappa) &= \pi + \int_{\infty}^{1} \frac{\kappa\, dr}{r(r^2 - \kappa^2)^{\frac{1}{2}}} + \int_{1}^{r^*} \frac{\kappa\, dr}{r[r^2n^2(r) - \kappa^2]^{\frac{1}{2}}} \\ &= \frac{3\pi}{4} - \sin^{-1}\kappa - \tfrac{1}{2}\tan^{-1}\frac{n_0^2 - 2\kappa^2}{2\kappa(1 - \kappa^2)^{\frac{1}{2}}}, \end{aligned} \tag{7}$$

on substituting (4) into the second integral and carrying out some straightforward integratious.

We can now determine the equation of the ray path inside the lens. Proceeding along the ray in either direction from the turning point, we have from (3) and (4),

$$\begin{aligned} \theta - \theta^*(\kappa) &= \pm \int_{r^*}^{r} \frac{\kappa \, dr}{r[n^2r^2 - \kappa^2]^{\frac{1}{2}}} \\ &= \pm \tfrac{1}{2} \cos^{-1} \frac{2\kappa^2 - n_0^2 r^2}{r^2[n_0^4 - 4\kappa^2(n_0^2 - 1)]^{\frac{1}{2}}}, \end{aligned} \tag{8}$$

or, by rearrangement,

$$F(r,\theta;\kappa) \equiv r^2\{[n_0^4 - 4\kappa^2(n_0^2 - 1)]^{\frac{1}{2}} \cos 2(\theta - \theta^*) + n_0^2\} - 2\kappa^2 = 0. \tag{9}$$

Since $\theta^*(\kappa)$ is constant for a particular ray, it is easy to write (9) in rectangular coordinates and to verify that it is the equation of an ellipse.

III. RATE OF INTERNAL DISSIPATION OF HEAT

If the total dissipated power is a small fraction of the incident power, as it must be in a practical lens, then the heat losses may be regarded as a small perturbation on the power flux in the lossless case, which we shall now compute.

Henceforth we regard each ray as defining a surface of revolution, although the ray itself lies in a plane through the axis of the system. According to geometrical optics, the total power flow along the tube bounded by the ray surfaces corresponding to κ and $\kappa + d\kappa$ is constant. Let $d\nu$ be the elementary distance normal to the ray in the direction of increasing κ. If κ is regarded as a point function defined by (9), we have

$$\frac{\partial \kappa}{\partial \nu} = \left| \frac{\partial F/\partial \nu}{\partial F/\partial \kappa} \right| = \left| \frac{\nabla F}{\partial F/\partial \kappa} \right|, \tag{10}$$

where ∇F is evaluated by differentiating F with respect to the coordinate variables while holding κ fixed.

Now let $S(r,\theta)$ be the power flux along a ray at any point of the lens; appropriate units for S with the present normalization of lengths are watts/(radius)^2. The total power flow along an elementary tube is then

$$dP = 2\pi r \sin \theta \, S(r,\theta) d\nu = \text{constant}. \tag{11}$$

The constant can be evaluated by considering a ring-shaped element of area normal to the incident beam, where κ is just the distance of the

given ray from the axis. If P_0 is the total power incident on the lens, then

$$dP = 2P_0\kappa \, d\kappa. \tag{12}$$

Combining (10), (11), and (12) yields

$$S(r,\theta) = \frac{P_0\kappa}{\pi r \sin\theta} \frac{|\nabla F|}{|\partial F/\partial\kappa|}. \tag{13}$$

The attenuation constant α at any point of the lens is given by the well-known approximate relationship

$$\alpha = (\pi n/\lambda) \tan\delta, \tag{14}$$

where n is the refractive index, $\tan\delta$ is the loss tangent (assumed small), and λ is the free-space wavelength, measured here in units of the lens radius. Hence the rate of energy dissipation per unit volume, in watts/(radius)^3, is

$$Q(r,\theta) = 2\alpha S(r,\theta) = \frac{2P_0\kappa n \tan\delta}{\lambda r \sin\theta} \frac{|\nabla F|}{|\partial F/\partial\kappa|}. \tag{15}$$

On the right side of (15), κ is defined implicitly by (7) and (9) as a function of r and θ. The refractive index $n(r)$ is given by (4), and $\tan\delta$ is supposed to be a known function of n. Differentiation of (7) and (9) yields

$$\frac{\partial F}{\partial r} = 2r\{[n_0^4 - 4\kappa^2(n_0^2 - 1)]^{\frac{1}{2}} \cos 2(\theta - \theta^*) + n_0^2\}, \tag{16}$$

$$\frac{1}{r}\frac{\partial F}{\partial\theta} = -2r\{[n_0^4 - 4\kappa^2(n_0^2 - 1)]^{\frac{1}{2}} \sin 2(\theta - \theta^*)\}, \tag{17}$$

$$\begin{aligned}\frac{\partial F}{\partial\kappa} = r^2 \Bigg\{ &-\frac{4\kappa(n_0^2 - 1)}{[n_0^4 - 4\kappa^2(n_0^2 - 1)]^{\frac{1}{2}}} \cos 2(\theta - \theta^*) \\ &+ 2[n_0^4 - 4\kappa^2(n_0^2 - 1)]^{\frac{1}{2}} \sin 2(\theta - \theta^*) \frac{d\theta^*}{d\kappa}\Bigg\} - 4\kappa,\end{aligned} \tag{18}$$

$$\frac{d\theta^*}{d\kappa} = -\frac{1}{(1 - \kappa^2)^{\frac{1}{2}}} \frac{(n_0^2 - 2\kappa^2)(n_0^2 - 1)}{n_0^4 - 4\kappa^2(n_0^2 - 1)}. \tag{19}$$

Hence in principle the rate of heat generation $Q(r,\theta)$ is a known function of position within the lens.

To determine the power flux from (13) and (16)–(19) when $\kappa = 0$ or $\kappa = 1$ requires the evaluation of some indeterminate forms. Derivations of the following results are straightforward and will be omitted.

The power flux along the axis is given by

$$S(\xi) = \frac{P_0}{\pi}\left[\frac{n_0^2}{[n_0^2 - (n_0^2 - 1)\xi^2]^{\frac{1}{2}} - (n_0^2 - 1)\xi}\right]^2, \tag{20}$$

where ξ is the normalized axial coordinate defined below (2) of Section I. The power flux at the equator of the lens is

$$\lim_{r\to 1^-} S(r,\pi/2) = P_0/\pi n_0^2. \tag{21}$$

If the focus is at the surface of the lens ($n_0 = \sqrt{2}$), then

$$\lim_{r\to 1^-} S(r,\theta) = 0, \qquad 0 < \theta < \pi/2. \tag{22}$$

In this case, particular interest attaches to the power flux in the immediate neighborhood of the focal point. It is convenient to introduce new polar coordinates ρ,ϑ, with origin at the focus and polar axis in the direction of decreasing ξ (see Fig. 1). Then for small ρ the power flux is

$$S \approx \frac{P_0 \cos\vartheta}{\pi\rho^2}. \tag{23}$$

IV. AXIAL TEMPERATURE DISTRIBUTION AND DISSIPATED POWER

The steady-state temperature distribution inside the lens satisfies Poisson's equation,

$$\nabla^2 T = -Q/k, \tag{24}$$

where T is the temperature above any convenient reference level, Q is the source distribution, and k is the thermal conductivity, expressed for the moment in units of watts/(degree·radius). Since the surface of the lens is assumed to be held at a constant temperature, say by air conditioning the space between the lens and the radome, the boundary condition may be taken as

$$T = 0 \quad \text{at} \quad r = 1. \tag{25}$$

The source distribution Q is a function of the coordinates r,θ only, and in all practical cases it decreases with increasing distance from the axis. It is proved in Appendix A that the maximum temperature rise then occurs on the axis; and as shown in Section I, a knowledge of the axial temperature distribution in this case enables us to put upper and lower bounds on the maximum temperature rise in a Luneberg lens illuminated from more than one direction.

The temperature distribution on the axis may easily be written down in terms of the Green's function for the interior of a sphere; thus,

$$T(\xi) = \frac{1}{k}\int_0^\pi \int_0^1 Q(r,\theta)G(r,\theta;\xi)2\pi r^2 \sin\theta \, dr \, d\theta, \tag{26}$$

where

$$G(r,\theta;\xi) = (1/4\pi)[(r^2 + \xi^2 - 2r\xi\cos\theta)^{-\frac{1}{2}} - (1 + \xi^2 r^2 - 2r\xi\cos\theta)^{-\frac{1}{2}}]. \tag{27}$$

If desired, one may think of (26) as representing the electrostatic potential due to a distributed electric charge of density $\epsilon_0 Q(r,\theta)/k$ inside an earthed, conducting sphere of unit radius, but the analogy has nothing to do with the mathematics.

Before starting calculation, we shall assume that $\tan\delta$ is a linear function of n, say

$$\tan\delta = An + B. \tag{28}$$

Then making use of (15), we may write (26) in the form

$$T(\xi) = (P_0/k\lambda)[AT_A(\xi) + BT_B(\xi)], \tag{29}$$

where

$$\begin{aligned} T_A(\xi) &= 4\pi\int_0^\pi \int_0^1 \kappa n^2 \left| \frac{\nabla F}{\partial F/\partial \kappa} \right| G(r,\theta;\xi) r \, dr \, d\theta, \\ T_B(\xi) &= 4\pi\int_0^\pi \int_0^1 \kappa n \left| \frac{\nabla F}{\partial F/\partial \kappa} \right| G(r,\theta;\xi) r \, dr \, d\theta. \end{aligned} \tag{30}$$

The dimensionless functions $T_A(\xi)$ and $T_B(\xi)$ are calculated numerically in the next section. Note that although the radius of the sphere has heretofore been taken as the unit of length, the factor $P_0/k\lambda$ has the dimensions of temperature, and any consistent set of units (e.g., MKS) may be used for P_0, k, and λ in (29).

It has been tacitly assumed in the foregoing that the thermal conductivity k is constant throughout the lens. But the conductivity of polystyrene foam, out of which Luneberg lenses are usually made, is known to increase with increasing temperature, and therefore it may be greater in some parts of the lens than in others. However we know from general theory that if in a body with a fixed distribution of heat sources, the thermal conductivity is increased at any point, the steady-state temperature at each point either decreases or remains unchanged. Hence the solution of the heat flow problem with a constant value of k

less than or equal to the actual value of k at all points, provides a temperature distribution which is an upper bound to the actual temperature distribution at all points.

Finally, the fraction of the total incident power which is dissipated in the lens is given by integrating Q/P_0 over the volume. We obtain, from (15) and (28),

$$\frac{\Delta P}{P_0} = \frac{a}{\lambda}\left[A\,\frac{\Delta P_A}{P_0} + B\,\frac{\Delta P_B}{P_0}\right], \tag{31}$$

where a is the radius of the lens in the same units as λ, and

$$\begin{aligned} \frac{\Delta P_A}{P_0} &= 4\pi \int_0^\pi \int_0^1 n^2\kappa \left| \frac{\nabla F}{\partial F/\partial \kappa} \right| r\,dr\,d\theta, \\ \frac{\Delta P_B}{P_0} &= 4\pi \int_0^\pi \int_0^1 n\kappa \left| \frac{\nabla F}{\partial F/\partial \kappa} \right| r\,dr\,d\theta. \end{aligned} \tag{32}$$

V. NUMERICAL RESULTS

The functions $T_A(\xi)$ and $T_B(\xi)$ were evaluated on an IBM 7090 by a straightforward double application of Simpson's rule to (30). The value of κ at each point was found by solving (9) by Newton's method; then $|\nabla F/(\partial F/\partial \kappa)|$ was calculated from (16)–(19) and $G(r,\theta;\xi)$ from (27). Two values of normalized focal distance, measured from the center of the lens, were considered, namely,

$$r_0 = 1.0, \qquad n_0 = \sqrt{2}, \tag{33}$$

$$r_0 = 1.1, \qquad n_0 = 1.36025. \tag{34}$$

The numerical results are given in Table I, and are plotted in Figs. 2 and 3. Also the total dissipated power was computed from (32). For $r_0 = 1.0$,

$$(\Delta P/P_0) = (a/\lambda)[16.91A + 13.97B]; \tag{35}$$

and for $r_0 = 1.1$,

$$(\Delta P/P_0) = (a/\lambda)[15.46A + 13.13B]. \tag{36}$$

For the numerical integration a graded net was used, as follows:

Region I	$r =$	0.00 (0.05) 0.40
	$\theta =$	0.0° (7.5°) 180.0°
Region II	$r =$	0.40 (0.05) 0.80
	$\theta =$	0.0° (3.0°) 12.0°

Region III $r = 0.80\ (0.02)\ 1.00$
$\theta = 0.0^\circ\ (1.0^\circ)\ 12.0^\circ$

Region IV $r = 0.40\ (0.05)\ 1.00$
$\theta = 12.0^\circ\ (3.0^\circ)\ 30.0^\circ$

Region V $r = 0.40\ (0.05)\ 1.00$
$\theta = 30.0^\circ\ (7.5^\circ)\ 180.0^\circ$

Experimentation with a finer net, obtained by simultaneously halving the intervals in r and θ in Regions III and V, indicates that any errors in Table I and (35)–(36) do not exceed a few units of the last figure shown. The accuracy is therefore believed to be sufficient for all practical purposes.

A few aspects of the calculation deserve comment. In the first place, the Green's function (27) is infinite when the field point coincides with the source point; but the integrand of (26) does not become infinite as the field point approaches the source point, provided that $Q(r,\theta)$ remains finite on the axis. The limiting value of the integrand may be either zero or finite, depending upon the direction from which the field point approaches the source point; but in any event the contribution of the apparent singular point during a naive application of Simpson's

Table I — The Functions $T_A(\xi)$ and $T_B(\xi)$

ξ	$r_0 = 1.0$		$r_0 = 1.1$	
	T_A	T_B	T_A	T_B
−1.0	0.0000	0.0000	0.0000	0.0000
−0.9	0.1060	0.0861	0.0985	0.0823
−0.8	0.2155	0.1727	0.1993	0.1645
−0.7	0.3241	0.2564	0.2981	0.2431
−0.6	0.4308	0.3369	0.3940	0.3179
−0.5	0.5345	0.4140	0.4860	0.3887
−0.4	0.6345	0.4876	0.5734	0.4553
−0.3	0.7298	0.5574	0.6553	0.5174
−0.2	0.8198	0.6234	0.7310	0.5749
−0.1	0.9037	0.6856	0.7998	0.6277
0.0	0.9809	0.7440	0.8609	0.6754
0.1	1.0493	0.7974	0.9123	0.7169
0.2	1.1090	0.8464	0.9535	0.7521
0.3	1.1588	0.8907	0.9831	0.7802
0.4	1.1984	0.9306	0.9997	0.8003
0.5	1.2267	0.9659	1.0005	0.8105
0.6	1.2386	0.9937	0.9785	0.8051
0.7	1.2317	1.0132	0.9255	0.7771
0.8	1.2040	1.0251	0.8243	0.7109
0.9	1.1521	1.0324	0.6206	0.5552
1.0	1.0000	1.0000	0.0000	0.0000

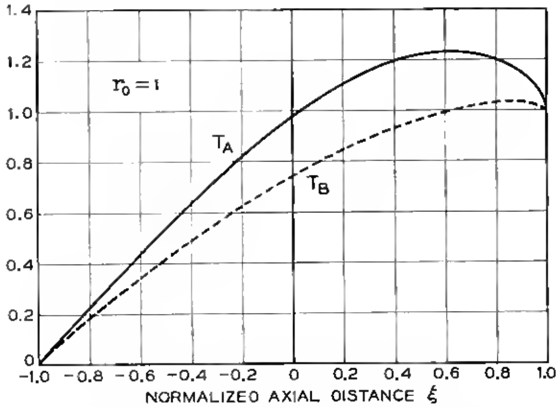

Fig. 2 — The functions T_A and T_B for a lens with focus at the surface ($r_0 = 1$).

rule is at most a finite quantity, which tends to zero as the interval of integration is reduced. In the numerical calculation the contribution from this point was omitted.

A discussion of the mathematical singularity at the focus when $r_0 = 1$ is given in Appendix B. In the mathematical model, the temperature near the focus is given by

$$T(\rho,\vartheta) \approx T_0 \cos \vartheta, \tag{37}$$

where T_0 is a finite constant and ρ,ϑ are spherical polar coordinates in the local coordinate system introduced at the end of Section III. The value of T_0 is determined by the source strength in an infinitesimal

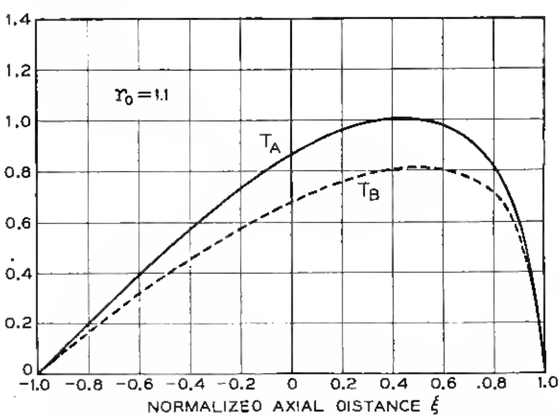

Fig. 3 — The functions T_A and T_B for a lens with focus outside the surface ($r_0 = 1.1$).

region around the focus; in the limit this region makes no contribution to the temperature at other points of the sphere. Hence $T_A(\xi)$ and $T_B(\xi)$ were calculated, for $\xi \neq 1$, by setting the integrands of (30) equal to zero at $r = 1$, $\theta = 0$; and $T_A(1)$ and $T_B(1)$ were evaluated as in Appendix B.

It goes without saying that in a physical lens the temperature at the focus will be well defined, which the limit of the expression (37) is not, and that the temperature distribution will not have an infinite gradient. The actual distribution will be determined by the amount of heat that the air conditioning can carry away from the immediate neighborhood of the focus, as well as by the physical structure of the feed, if the antenna is being used for transmission. Since, however, the numerical example at the end of this section suggests that the maximum temperature rise in a Luneberg lens with surface cooling will be well inside the lens, we shall not attempt here a more elaborate analysis of the conditions near the focus.

If $r_0 > 1$, so that the focal point is outside the lens surface, then (4) is not an exact expression for the index of refraction. However, when $r_0 = 1.1$, the maximum difference between the exact index calculated according to Ref. 2 and the approximate index is about 0.0055 at about $r = 0.87$, the approximate index being smaller. As a second test, we have calculated the distance from the center of the "approximate" lens at which various initially parallel rays intersect the axis. The distance varies from 1.0881 for paraxial rays ($\kappa = 0$) to 1.1205 for rays with $\kappa = 0.95$, compared with the design value of 1.1. It tends toward infinity for marginal rays, but such rays are insignificant so far as the heating problem is concerned anyway, since by hypothesis the surface of the lens is in contact with a constant-temperature heat reservoir. We therefore feel well justified in using (4) to compute the ray paths for $r_0 = 1.1$.

To give an idea of the size of the numbers involved, Fig. 4 shows plots of the axial temperature rise in degrees Fahrenheit per watt of incident power, as calculated from (29) and Table I for a lens with the following parameters:

$$\begin{aligned} P_0 &= 1 \text{ watt} \\ k &= 0.25 \frac{\text{BTU}}{\text{hr}\cdot\text{ft}^2\cdot(^\circ\text{F/in})} = 0.036 \frac{\text{watts}}{\text{m}\cdot{}^\circ\text{C}} \\ \lambda &= 60 \text{ cm } (500 \text{ mc}) \\ \tan\delta &= [1 + 25(n-1)] \times 10^{-4} \\ a &= 40 \text{ ft} \end{aligned} \tag{38}$$

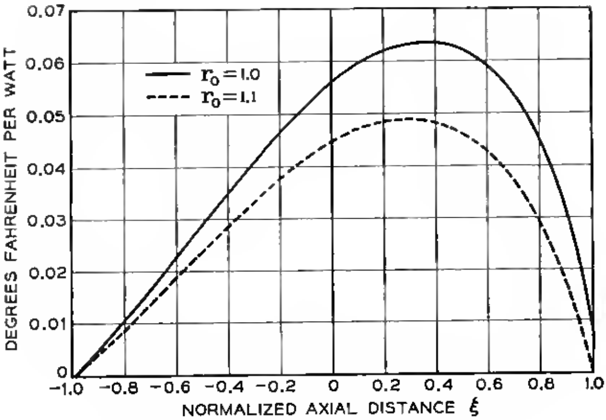

Fig. 4 — Numerical example of axial temperature distribution in two Luneberg lenses.

The assumed values of thermal conductivity and loss tangent are more or less representative of polystyrene foam loaded with metal slivers. The maximum temperature rise is roughly 0.06°F/watt, and occurs about a third of the way from the center of the lens to the surface, in the direction of the focal point. It is of interest to note that the maximum temperature differs from the temperature at the center of the lens by less than 15 per cent, even when the focus is at the surface.

The power dissipated in the lens is easily calculated from (35) or (36). For $r_0 = 1.0$,

$$(\Delta P/P_0) = 0.1777 \quad \text{or} \quad 0.85 \text{ db}; \tag{39}$$

while for $r_0 = 1.1$,

$$(\Delta P/P_0) = 0.1450 \quad \text{or} \quad 0.68 \text{ db}. \tag{40}$$

Equations (39) and (40) give the dissipated power when a uniform plane wave is incident on the lens. Observe, however, that this is not quite equal to the power loss when the lens is being used as a transmitter, since in that case there is usually a deliberate illumination taper across the lens aperture. With a conventional taper, in which the power density is higher at the center of the lens than at the edges, the loss will be higher than that obtained with uniform illumination; but one cannot deduce the total loss from the numbers given in this paper.

VI. ACKNOWLEDGMENTS

This work has had the benefit of valuable discussions with J. A. Lewis and W. A. Yager. The IBM programs were written by Mrs. Marie Dolan.

Note Added in Proof

A recent paper by Lerner[4] is concerned with essentially the same problem as the present paper. On the basis of a considerable number of approximations, Lerner calculates the temperature distribution along the axis of a surfaee-focus lens when the thermal conductivity and the loss tangent are constant. The "Average" curve of his Fig. 6 is therefore comparable to the plot of $T_B(\xi)$ in our Fig. 2. The two curves do in fact lie close together except for points less than a quarter of the lens radius distant from the surface focus. Lerner's approximate analysis predicts that the maximum temperature rise will occur at the surface focus and will be equal to $1.25(P_0/k\lambda)\tan\delta$. Our calculations give a maximum rise of about $1.03(P_0/k\lambda)\tan\delta$ at a distance of about one-tenth of the lens radius from the focus, while the temperature rise at the focus is $(P_0/k\lambda)\tan\delta$.

APPENDIX A

Position of Temperature Maximum

We consider the steady-state temperature distribution which satisfies

$$\begin{aligned} \nabla^2 T &= -f \quad \text{for} \quad r < a, \\ T &= 0 \qquad \text{at} \quad r = a. \end{aligned} \tag{41}$$

The source function f $(= Q/k)$ is assumed to be axially symmetric and nonnegative, and to have continuous first derivatives in the region $r < a$.

We shall use rectangular coordinates (x,y,z), cylindrical coordinates $(\mathfrak{r},\varphi,z)$, or spherical coordinates (r,θ,φ) as convenient. It is assumed that f is independent of φ, and that it is a nonincreasing function of distance from the axis, i.e.,

$$(\partial f/\partial \mathfrak{r}) \leqq 0. \tag{42}$$

Now consider the function

$$W \equiv -(\partial T/\partial y) \equiv -(\partial T/\partial \mathfrak{r})\sin\varphi \tag{43}$$

in the hemispherical region S defined by

$$\mathfrak{r}^2 + z^2 \leqq a^2, \qquad 0 \leqq \varphi \leqq \pi. \tag{44}$$

From (41) and (42) it follows that W is superharmonic in the interior of S, since

$$\nabla^2 W = (\partial f/\partial y) = (\partial f/\partial \mathfrak{r}) \sin \varphi \leqq 0. \tag{45}$$

Furthermore on the curved surface of the hemisphere we have

$$W = -(\partial T/\partial y) = -(y/r)(\partial T/\partial r) > 0 \quad \text{for} \quad r = a, \quad y > 0, \tag{46}$$

since

$$(\partial T/\partial r) < 0 \quad \text{at} \quad r = a \tag{47}$$

if f is nonnegative and does not vanish identically. A formal proof of this physically obvious statement can be obtained using the Green's function representation of the solution of (41). On the base of the hemisphere, (43) yields

$$W = 0 \quad \text{at} \quad y = 0. \tag{48}$$

Let U be a harmonic function which takes the same values as W on the boundary of S. Since W is superharmonic,[3] we have

$$W \geqq U \tag{49}$$

in the interior of S. But U achieves its minimum value zero only on the boundary of S, so neither U nor W can vanish in the interior of S. It follows from (43) that $\partial T/\partial \mathfrak{r}$ cannot vanish in the interior of S, and so T cannot have a maximum there. Hence the maximum value of T must occur on the axis.

In the present problem the source distribution is given by (15), namely

$$f(r,\theta) = \frac{Q(r,\theta)}{k} = \frac{2P_0 \tan \delta}{\lambda k} \left\{ \frac{\kappa n}{r \sin \theta} \left| \frac{\nabla F}{\partial F/\partial \kappa} \right| \right\}. \tag{50}$$

An analytic proof that the right-hand side of (50) is a decreasing function of distance from the axis would probably be very laborious. We have, however, calculated the expression in braces numerically for the two cases treated in this paper, using a square grid of about 600 points in $\mathfrak{r}$ and z, and have verified that on such a grid it is a decreasing function of $\mathfrak{r}$, except for a very small region near the surface of the lens (where $\mathfrak{r}$ is slightly less than 1 and θ slightly greater than $\pi/2$ in Fig. 1). On the other hand, the refractive index n is a decreasing function of $\mathfrak{r}$,

and in practical cases tan δ will be a sufficiently rapidly increasing function of n so that the whole source distribution will be a decreasing function of $\mathfrak{r}$. In particular, we have verified numerically that (42) is satisfied throughout the lens if tan δ is given by (38).

It will be appreciated, of course, that (42) is a sufficient condition but not a necessary one to make the maximum temperature rise occur on the axis. Whether there could exist a hypothetical loss tangent, having a sharp peak at a particular value of n, which would lead to a ring-shaped temperature maximum instead of to a maximum on the axis is still an open question, though not of much practical importance.

APPENDIX B

Temperature Distribution near a Surface Focus

To determine the nature of the temperature singularity in the present mathematical model at a surface focus, we investigate the temperature distribution $T(\rho,\vartheta)$ in the half-space $\vartheta \leqq \pi/2$ due to the source function

$$Q(\rho,\vartheta) = \begin{cases} \dfrac{C \cos \vartheta}{\rho^2}, & 0 < \rho < b, \\ 0, & \rho \geqq b. \end{cases} \tag{51}$$

Here b and C are constants, and ρ,ϑ are the polar coordinates introduced at the end of Section III. All quantities are independent of the azimuth angle φ. We seek a solution which vanishes on the plane $\vartheta = \pi/2$, remains finite as $\rho \to 0$ and as $\rho \to \infty$, and is continuous, together with its normal derivative, at $\rho = b$.

Substitution of a function of the form

$$T(\rho,\vartheta) = R(\rho) \cos \vartheta \tag{52}$$

into Poisson's equation (24) yields the following equation for $R(r)$:

$$\frac{d}{dr}\left(\rho^2 \frac{dR}{d\rho}\right) - 2R = \begin{cases} -C/k, & 0 < \rho < b, \\ 0, & \rho \geqq b. \end{cases} \tag{53}$$

A solution which satisfies the boundary and continuity conditions is easily found to be

$$\begin{aligned} R(\rho) &= \frac{C}{2k}\left(1 - \frac{2\rho}{3b}\right), & 0 < \rho < b, \\ R(\rho) &= \frac{C}{6k}\frac{b^2}{\rho^2}, & \rho \geqq b. \end{aligned} \tag{54}$$

We observe that

$$\lim_{\rho \to 0} R(\rho) = C/2k, \tag{55}$$

and this limit is independent of b. On the other hand, if ρ has any fixed value greater than zero,

$$\lim_{b \to 0} R(\rho) = 0. \tag{56}$$

It follows that the limiting value of the axial temperature is determined by the source distribution in an arbitrarily small hemisphere around $\rho = 0$, and that in the limit this hemisphere makes no contribution to the temperature at other points of the lens.

If we make the constant C agree with the source distribution given by (15) and (23), then using (28) and the fact that $n = 1$ at the surface of the lens, we have

$$C = \frac{2\alpha P_0}{\pi} = \frac{2(A + B)P_0}{\lambda}. \tag{57}$$

Combining (57) with (55) leads to the results given in Table I, namely

$$\lim_{\xi \to 1^-} T_A(\xi) = \lim_{\xi \to 1^-} T_B(\xi) = 1. \tag{58}$$

APPENDIX C

Time Scale for Transient Heating

Suppose that in a Luneberg lens, initially at zero temperature throughout, an internal source distribution $Q(r,\theta)$ is turned on at $t = 0$ and then remains constant in time, while the surface of the lens is held at zero temperature. The temperature within the lens must satisfy

$$k\nabla^2 T + Q = \rho c(\partial T/\partial t), \tag{59}$$

where k is the thermal conductivity, ρ the density, and c the specific heat of the medium. Conventional units, such as MKS, are used throughout this section.

It is well known that the solution of (59) can be written as the sum of a steady-state part and a transient part, i.e.,

$$T(r,\theta,t) = T_s(r,\theta) + T_t(r,\theta,t). \tag{60}$$

The steady-state term satisfies

$$k\nabla^2 T_s = -Q, \tag{61}$$

while the transient term satisfies

$$D\nabla^2 T_t = \partial T_t / \partial t, \tag{62}$$

where

$$D = k/\rho c \tag{63}$$

is the diffusivity. The transient term vanishes as $t \to \infty$ and cancels the steady-state solution at $t = 0$; that is,

$$T_t(r,\theta,0) = -T_s(r,\theta). \tag{64}$$

At the surface of the sphere both terms vanish; namely,

$$T_s(a,\theta) = T_t(a,\theta,t) = 0. \tag{65}$$

A transient solution sufficiently general for our needs may be written in the form

$$T_t(r,\theta,t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{nm} j_n(\chi_{nm} r/a) P_n(\cos\theta) e^{-\alpha_{nm} t}, \tag{66}$$

where j_n is a spherical Bessel function defined in terms of the ordinary Bessel function by

$$j_n(x) = (\pi/2x)^{\frac{1}{2}} J_{n+\frac{1}{2}}(x), \tag{67}$$

P_n is a Legendre polynomial, χ_{nm} is the mth root of $j_n(x)$, and

$$\alpha_{nm} = D\chi_{nm}^2/a^2. \tag{68}$$

The Bessel and Legendre functions form complete, orthogonal sets, so that at $t = 0$ any reasonable function of r and θ may be expanded in a double series of the form (66), where the coefficients A_{nm} are given by integrals similar to those which define the coefficients in a double Fourier series. In particular, if we had calculated the steady-state solution $T_s(r,\theta)$ at all points of the sphere, we could expand it in such a series and thus satisfy the initial condition (64). We shall not compute the coefficients A_{nm}; we merely observe that in a transient solution of the form (66), the individual terms decay exponentially with time, the faster the larger α_{nm}. The longest-lived term is the one with smallest α, namely

$$A_{01} j_0(\chi_{01} r/a) e^{-\alpha_{01} t} = A_{01} \frac{\sin(\pi r/a)}{\pi r/a} e^{-(D\pi^2/a^2)t}. \tag{69}$$

The rate at which this term decays permits us to define an approximate "thermal time constant" for the transient solution, unless of course the steady-state solution is such that A_{01} is zero or very small compared to the other coefficients. Comparing the form of (69) with the expected form of the steady-state solution makes it appear obvious that A_{01} will not be unusually small; and therefore an estimate of the time required to establish the steady-state solution is furnished by the "half-life" of the lowest mode,

$$t_{01} = 1/\alpha_{01} = a^2/\pi^2 D. \tag{70}$$

Assuming for polystyrene foam the numerical values

$$\begin{aligned} \rho &= 1.5 \text{ lb/ft}^3, \\ c &= 0.32 \text{ cal/gm}\cdot{}^\circ\text{C}, \\ k &= 0.25 \frac{\text{BTU}}{\text{hr}\cdot\text{ft}^2\cdot({}^\circ\text{F/in})}, \end{aligned} \tag{71}$$

we find after some conversions of units,

$$D = 1.12 \times 10^{-6} \text{ m}^2/\text{sec}. \tag{72}$$

For a sphere of diameter 1 ft,

$$t_{01} = 35 \text{ min}, \tag{73}$$

and for a sphere of diameter 80 ft,

$$t_{01} = 156 \text{ days}. \tag{74}$$

The heating time for a large Luneberg lens may thus be several months after the transmitter is turned on, with a similar cooling time after it is turned off.

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